

Uniqueness of Tangent Cones to Positive- (p, p) Integral Cycles

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Abstract: *Let (\mathcal{M}, ω) be a symplectic manifold, endowed with a compatible almost complex structure J and the associated metric g . For any $p \in \{1, 2, \dots, \frac{1}{2} \dim \mathcal{M}\}$ the form $\Omega := \frac{1}{p!} \omega^p$ is a calibration. More generally, dropping the closedness assumption on ω , we get an almost hermitian manifold $(\mathcal{M}, \omega, J, g)$ and then Ω is a so-called semi-calibration. We prove that integral cycles of dimension $2p$ (semi-)calibrated by Ω possess at every point a unique tangent cone. The argument relies on an algebraic blow up perturbed in order to face the analysis issues of this problem in the almost complex setting.*

1 Introduction

The notion of calibration appeared in the foundational paper [10] in 1982, after that key features of calibrations had been observed in some particular cases in the previous decades (see [15] for a historical overview).

An immediate impact of calibrated currents was in connection with Plateau's problem, since these objects are mass-minimizers in their homology class and thus provide plenty of interesting and explicit examples of volume-minimizers. In the last fifteen years, however, calibrations have appeared surprisingly in many other geometric or physical problems, for example (see [6], [21], [22], [23], [24]) theory of invariants, Yang-Mills fields, String theory, etc. Typically an essential issue in these studies is to understand regularity properties of calibrated currents.

Already raised in [10], one of the long-standing regularity questions is whether calibrated integral currents admit unique tangent cones. The issue is still open, except for currents of dimension 2. Let us recall a few notions and the state of the art, before passing to the results of the present work.

Given a m -form ϕ on a Riemannian manifold (M, g) , the comass of ϕ is defined to be

$$||\phi||^* := \sup\{\langle \phi_x, \xi_x \rangle : x \in M, \xi_x \text{ is a unit simple } m\text{-vector at } x\}.$$

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A form ϕ of comass one is called a *calibration* if it is closed ($d\phi = 0$). We will be dealing also with non-closed forms of unit comass, which will be referred to as *semi-calibrations*, following the terminology in [16].

Let ϕ be a (semi)calibration; among the oriented m -dimensional planes of the Grassmannians $G(m, T_x M)$, we pick those on which ϕ agrees with the m -dimensional volume form. Representing oriented m -dimensional planes as unit simple m -vectors, we are thus selecting the subfamily of the so-called *m -planes calibrated by ϕ* :

$$\mathcal{G}(\phi) := \cup_{x \in M} \{\xi_x \in G(m, T_x M) : \langle \phi_x, \xi_x \rangle = 1\}.$$

An integral current of dimension m is said to be ϕ -(semi)calibrated if, \mathcal{H}^m -almost everywhere, its approximate tangent planes belong to \mathcal{G} . A simple argument (see [10]) then shows that calibrated currents are homologically mass-minimizing, while semi-calibrated ones are almost minimizers (or λ -minimizers, using the terminology of [7]).

Examples of well-known calibrations are the symplectic form ω in an almost Kähler manifold, its normalized powers $\frac{1}{p!}\omega^p$, the Special Lagrangian calibration in Calabi-Yau m -folds, the Associative calibration, and many others.

If we drop the closedness assumption on ω in the definition of almost Kähler manifold, we get what is usually called an almost Hermitian manifold, and ω and $\frac{1}{p!}\omega^p$ are then semicalibrations. We will refer to these as almost-complex semi-calibrations.

When dealing with a boundaryless integral current C , also called integral cycle, or simply when we localize the current to an open set in which the boundary is zero, it turns out that calibrated currents satisfy an important *monotonicity* formula for the mass ratio: for any x_0 , the quantity $\frac{M(C \llcorner B_r(x_0))}{r^m}$ is a weakly increasing function of r . This is a classical result for mass-minimizers (see [8], [19] or [15]), proved for constant calibrations in [10].

When we turn our attention to almost-minimizers, what we get is an *almost-monotonicity* formula, see [7] and [19]. Almost-monotonicity was proved for C^1 semi-calibrations in [16]: it states that the mass ratio $\frac{M(C \llcorner B_r(x_0))}{r^m}$ is given by a weakly increasing function of r plus a perturbation term, that is infinitesimal of r . The perturbation term is bounded in modulus independently of x_0 .

Immediate consequences of (almost) monotonicity are:

- (i) the *density* of the current is well-defined for every point x_0 as the limit

$$\nu(x_0) := \lim_{r \rightarrow 0} \frac{M(C \llcorner B_r(x_0))}{\alpha_m r^m},$$

where α_m is the m -dimensional volume¹ of the unit ball B^m .

- (ii) the density is an *upper semi-continuous* function.
- (iii) the density of a semi-calibrated integral cycles is, everywhere on the support², bounded by 1 from below³.

Monotonicity further yields the existence of *tangent cones*: this is a first step in the study of regularity of calibrated currents. The notion of tangent cone to a current C at a point x_0 is defined by the following procedure, called the *blow up limit*, whose idea goes back to De Giorgi [5]. Dilate C around x_0 of a factor r ; in normal coordinates around x_0 this amounts to pushing forward C via the map $\frac{x - x_0}{r}$:

$$(C_{x_0, r} \llcorner B_1)(\psi) := \left[\left(\frac{x - x_0}{r} \right)_* C \right] (\chi_{B_1} \psi) = C \left(\chi_{B_r(x_0)} \left(\frac{x - x_0}{r} \right)^* \psi \right). \quad (1)$$

The fact that $\frac{M(C \llcorner B_r(x_0))}{r^2}$ is monotonically almost-decreasing as $r \downarrow 0$ gives that, for $r \leq r_0$ (for a small enough r_0), we are dealing with a family of cycles $\{C_{x_0, r} \llcorner B_1\}$ in B_1 that are equibounded in mass. Therefore Federer-Fleming's compactness theorem (see e.g. [9] page 141) gives that there exist weak limits of $C_{x_0, r}$ as $r \rightarrow 0$. Every such limit C_∞ is an integer multiplicity rectifiable boundaryless current which turns out to be a cone⁴ calibrated by ω_{x_0} and is called a tangent cone to C at x_0 . The density of each tangent cone at the vertex is the same as the density of C at x_0 (see [10]).

The natural first question, raised already in [10], is whether Federer-Fleming's compactness theorem can yield different sequences of radii with different cones as limits, i.e. whether the tangent cone at an arbitrary point is *unique* or not. The answer is positive for semi-calibrated integral cycles of dimension 2, as proved in [16]. The uniqueness is also known for mass-minimizing integral currents of dimension 2, thanks to [25]. In some other

¹Recall that an arbitrary integral current C is defined by assigning on an oriented m -rectifiable set \mathcal{C} an integer-valued *multiplicity function* $\theta \in L^1(\mathcal{C}, \mathbb{N})$. For an arbitrary integral current, the density ν is well-defined \mathcal{H}^m -a.e. and agrees \mathcal{H}^m -a.e. with θ . What we get for semicalibrated cycles is that the density ν is well-defined everywhere, and we can take ν as the "precise representative" for the multiplicity θ .

²The support of a current C is the complement of the largest open set in which the action of the current is zero.

³This fails for arbitrary integral currents, as the example of the current of integration on a cone (counted with multiplicity 1) shows: the density at the vertex, although well-defined, depends on the opening angle of the cone (the narrower the cone is, the lower the density is). If we take, instead of a cone, a surface with a cusp point, the density there is 0.

⁴A current is said to be a cone with vertex p if it is invariant under omotheties centered at p .

cases, which also follow from either of the forementioned [25] or [16], the proof has been achieved using techniques of positive intersection, namely for integral pseudo-holomorphic cycles in dimension 4 ([22], [17]) and for integral Special Legendrian cycles in dimension 5 ([2], [3]). In [18] the uniqueness for pseudo holomorphic integral 2-dimensional cycles is achieved in arbitrary codimension. In [20] it is proved that if a tangent cone to a minimal integral current has multiplicity one and has an isolated singularity, then it is unique.

In dimensions higher than two, apart from L. Simon's result [20], the uniqueness of tangent cones is a completely open question. In this work we give a positive answer in the case of pseudo-holomorphic integral currents (i.e. semi-calibrated by the almost complex semi-calibration $\frac{1}{p!}\omega^p$) of arbitrary dimension and codimension.

The uniqueness cannot be obtained merely as a consequence of the monotonicity of the mass ratio. Indeed, the notion of being calibrated by ω can be extended from integral currents to normal ones (then it is usually called ω -positiveness, as in [10]). Normal positive cycles still fulfil the same monotonicity formula, but it was proved in [13] that they might have non-unique tangent cones.

The approach presented in this work relies on an algebraic blow up technique, adapted to the almost complex setting, that shows how, for almost-complex semi-calibration, uniqueness of tangent cones can indeed be obtained for integral semi-calibrated cycles just as a consequence of almost monotonicity and of the fact that the density is bounded by 1 from below.

The same approach was used in [4] for pseudo-holomorphic non-rectifiable currents of dimension 2. Here we extend that technique to higher dimensional pseudo-holomorphic currents: these are also referred to as *positive-(p, p) currents*, as we will see in the next section, where we describe more closely the setting and the result.

2 The main result

Let (\mathcal{M}, J) be an almost complex manifold of dimension $2n + 2$, where J is an almost complex structure. Let ω be a non-degenerate differential form of degree 2 compatible with J . If $d\omega = 0$ then we have a symplectic form, but we will not need to assume closedness. Let g be the Riemannian metric defined by $g(\cdot, \cdot) := \omega(\cdot, J\cdot)$. The triple (\mathcal{M}, J, g) is an almost Hermitian manifold; when the associated form ω is closed, we get an almost Kähler manifold. The word “almost” refers to the fact that J can be non-integrable.

On the other hand, given a constant non-degenerate differential 2-form ω in a $(2n + 2)$ -dimensional vector space V with a scalar product, if ω has unit comass then the non-zero form ω^{n+1} is the volume form and (V, ω, g) is a symplectic vector space with a compatible metric. So given any Riemannian

manifold (\mathcal{M}, g) (where g is the metric), if ω is a non-degenerate two-form with pointwise unit comass, then the non-zero form ω^{n+1} is the volume form and, for any $x \in \mathcal{M}$, $(T_x \mathcal{M}, \omega, g)$ is a symplectic vector space with a compatible metric. Then we can define an almost complex structure J on the tangent bundle of \mathcal{M} by setting $J := g^{-1}\omega$.

It is therefore equivalent for the purposes of this work to start with an almost complex structure J or with a semicalibration ω having pointwise unit comass. We can always assume to have a compatible triple (ω, g, J) on \mathcal{M} .

For any fixed integer $p \in \{1, 2, \dots, n\}$ the form $\Omega := \frac{1}{p!}\omega^p$ is a semi-calibration on \mathcal{M} for the metric g , i.e. the comass $\|\Omega\|^*$ is 1. This is nothing else but Wirtinger's inequality ([26]). If ω is closed, then so is Ω and we get a calibration.

Recall that the family $\mathcal{G}(\Omega)$ of $2p$ -planes calibrated by Ω is

$$\mathcal{G}(\Omega) := \cup_{x \in \mathcal{M}} \mathcal{G}_x := \cup_{x \in \mathcal{M}} \{\xi_x \in G_{2p}(x, T_x \mathcal{M}) : \langle \Omega_x, \xi_x \rangle = 1\}.$$

By Wirtinger's theorem [26] \mathcal{G}_x is made exactly of the $2p$ -dimensional J_x -complex subspaces of $T_x \mathcal{M}$. For this reason the $2p$ -planes in $\mathcal{G}(\Omega)$ are also called **positive**-(p, p) vectors. It is worthwhile noticing that the property of being a complex subspace of $(T_x \mathcal{M}, J_x)$ is not affected by choosing different couples (g_x, ω_x) and (g'_x, ω'_x) compatible with J_x in $T_x \mathcal{M}$. Therefore a positive-(p, p) vector is calibrated both by $\Omega := \frac{1}{p!}\omega^p$ in $(T_x \mathcal{M}, g)$ and $\Omega' := \frac{1}{p!}\omega'^p$ in $(T_x \mathcal{M}, g')$. This observation is of key importance for our proof.

We will consider a $2p$ -dimensional integral cycle T (semi)-calibrated by Ω . This means that \vec{T} (the oriented approximate tangents to T) belong \mathcal{H}^{2p} -a.e. to $\mathcal{G}(\Omega)$, or equivalently that $T(\Omega) = \int_{\mathcal{M}} \langle \Omega, \vec{T} \rangle d\|T\| = M(T)$.

The absence of boundary means that it holds, for any compactly supported $(2p-1)$ -form α , $(\partial C)(\alpha) := C(d\alpha) = 0$.

The issues we will be dealing with, namely tangent cones, are local: we will be only interested in the asymptotic behaviour of currents around a point, so we can assume to work in a chart rather than on a manifold. Ω -positive normal cycles in \mathbb{R}^{2n+2} satisfy the following important *almost monotonicity property* for the mass-ratio at any point x_0 .

Proposition 1. *Let \mathbb{R}^{2n+2} be endowed with a Riemannian metric g and a non-degenerate two-form ω of unit comass. Denote by Ω the semi-calibration $\frac{1}{p!}\omega^p$. Let the $2p$ -dimensional normal cycle T be Ω -positive and let x_0 be an arbitrary point. Denote by $B_r(x_0)$ the geodesic ball around x_0 of radius r .*

For an arbitrarily chosen point x_0 , the mass ratio $\frac{M(T \llcorner B_r(x_0))}{r^{2p}}$ is an almost-increasing function in r , i.e. $\frac{M(T \llcorner B_r(x_0))}{r^{2p}} = R(r) + O(r)$ for a

function R which is monotonically non-increasing as $r \downarrow 0$ and a function $O(r)$ which is infinitesimal.

This is proved in [16], Proposition 1. It is important to notice that the same proof works if we assume the semi-calibration to be just Lipschitz continuous rather than C^1 , see the appendix of [4]. The perturbation term $O(r)$ is bounded, independently of x_0 , by $C \cdot L \cdot r$, where C is a dimensional constant and L is the Lipschitz constant of the semi-calibration.

In this work we prove:

Theorem 2.1. *Given an almost complex $(2n + 2)$ -dimensional manifold $(\mathcal{M}, J, \omega, g)$, with a non-degenerate two-form ω compatible with J and associated Riemannian metric g , denote by Ω the form $\Omega := \frac{1}{p!} \omega^p$, for a fixed $p \in \{1, 2, \dots, n\}$; let T be a positive- (p, p) integral cycle, i.e. an integral $2p$ -cycle semi-calibrated by Ω .*

Then for any x_0 the tangent cone to T at x_0 is unique.

Theorem 2.1 reduces to the result in [16] in the case $p = 1$ and follows from [25] if $p = 1$ and $d\omega = 0$.

With a suitable choice of coordinates we can identify the tangent space $T_{x_0}\mathcal{M}$, endowed with the complex structure J_{x_0} , with \mathbb{C}^{n+1} : then every tangent cone T_∞ to T at x_0 is a positive- (p, p) cone in \mathbb{C}^{n+1} : such a cone is uniquely defined by a holomorphic $(p - 1, p - 1)$ integral cycle L_∞ in \mathbb{CP}^n .

Using the regularity theory for holomorphic integral cycles ([12], [11], [1]) we can deduce that L_∞ is in fact the sum of a finite number of holomorphic algebraic varieties⁵, each one taken with a constant integer multiplicity, but we will not need this result.

We will prove first the following

Lemma 2.1. *Let T be a Ω -semi-calibrated integral cycle and x_0 an arbitrary point. Then all tangent cones to T at x_0 have a uniquely determined support.*

Once this lemma is achieved, the uniqueness of tangent cones (i.e. theorem 2.1) follows with a few extra considerations (without making use of the results in [12], [11], [1]) developed in section 4, namely: **(i)** the space of tangent cones to T at x_0 is closed and connected in the space of $2p$ -integral cycles, **(ii)** the density is continuous under convergence of calibrated integral cycles sharing the same support.

As for lemma 2.1, the key idea for its proof is the analysis implementation, in the almost complex setting in which we are working, of the classical algebraic blow up. This was already used in [4] for positive- $(1, 1)$ normal cycles

⁵We are slightly abusing language here: these are algebraic varieties that are holomorphic away from their possible singular set.

and is here generalized to higher dimensional pseudo-holomorphic currents. The technique clearly shows that the uniqueness of theorem 2.1 holds just for “density reasons” (recall that the uniqueness can fail when we look at non-rectifiable currents, where the density is allowed to take any values ≥ 0 , see [13] and [4]).

3 Strategy and tools for the proof of the theorem

The first important remarks are contained in the following

Lemma 3.1. *Let T be as in theorem 2.1 and be \mathcal{T} the support of T . Assume that there exists a sequence of points $x_m \in \mathcal{T}$ with $x_m \rightarrow x_0$ and $x_m \neq x_0$ such that $\frac{x_m - x_0}{|x_m - x_0|} \rightarrow y \in S^{2n+1}$. Then there exists a tangent cone to T at x_0 , say T_∞ , such that the point y belongs to the support of T_∞ .*

On the other hand, if $y \in S^{2n+1}$ belongs to the support of a tangent cone T_∞ to T at x_0 , then there exists a sequence $x_m \rightarrow x_0$ (with $x_m \neq x_0$) of points x_m in the support of T such that $\frac{x_m - x_0}{|x_m - x_0|} \rightarrow y$.

Remark 3.1. Let $C_k \rightharpoonup C_\infty$ be a sequence of ϕ_k -semi-calibrated integral cycles ($k \in \mathbb{N} \cup \{\infty\}$), where ϕ_k are semi-calibrations with respect to the metrics g_k , and assume that the ϕ_k converge uniformly to ϕ_∞ , g_k converge uniformly to g_∞ and the C_k ’s have equibounded masses. Then $M(C_k \llcorner B) \rightarrow M(C_\infty \llcorner B)$ for any open set B . This follows because computing the mass for a semicalibrated current amounts to testing the current on the semi-calibration, so the convergence of the masses follows from the definition of weak-convergence of currents.

Remark 3.2. Recall that, as a consequence of monotonicity, a point belongs to the support of a semi-calibrated integral cycle if and only if its density is ≥ 1 .

proof of lemma 3.1. The first statement follows by choosing the sequence of radii $r_m := |x_m - x_0|$ and by looking at the sequence T_{x_0, r_m} . Up to a subsequence we may assume that $T_{x_0, r_m} \rightharpoonup T_\infty$. Each x_m is of density ≥ 1 for T by assumption and, for any m , the point $\frac{x_m - x_0}{|x_m - x_0|}$ is of density ≥ 1 for T_{x_0, r_m} . Since $\frac{x_m - x_0}{|x_m - x_0|} \rightarrow y$, analogously to remark 3.1 we can get

$$M\left(T_{x_0, r_m} \llcorner B_R\left(\frac{x_m - x_0}{|x_m - x_0|}\right)\right) \rightarrow M(T_\infty \llcorner B_R(y))$$

for any $R > 0$. By the almost monotonicity formula $M(T_\infty \llcorner B_R(y)) \geq \alpha_{2p} R^{2p}$ and so y is a point of density ≥ 1 for T_∞ .

Let now $y \in S^{2n+1}$. If there exists no sequence $x_m \neq x_0$ such that $x_m \in \mathcal{T}$, $x_m \rightarrow x_0$ and $\frac{x_m - x_0}{|x_m - x_0|} \rightarrow y$, then we can assume to have a ball $B_a^{2n+1}(y) \subset S^{2n+1}$ such that the cone $0 \# B_a^{2n+1}(y)$ is disjoint from

$\mathcal{T} \cap B_R^{2n+2}(0)$, for some small $R > 0$. But then, for any dilation $T_{x_0, r}$ with $r < R$ we have $M(T_{x_0, r} \llcorner B_a^{2n+2}(y)) = 0$. Since the mass passes to the limit for convergence of semi-calibrated cycles (remark 3.1), we deduce that y is a point of density 0 for any limit of the family $T_{x_0, r}$, therefore it cannot appear as a point in the support of any tangent cone. \square

In order to achieve lemma 2.1, it suffices, thanks to lemma 3.1, to analyze limits of $\frac{x_m - x_0}{|x_m - x_0|} \rightarrow y$ for $x_m \in \mathcal{T}$, $x_m \rightarrow x_0$. More precisely, recalling that each tangent cone is a holomorphic (p, p) -cone, if $y \in S^{2n+1}$ belongs to the support of a tangent cone T_∞ , then every point in the Hopf fiber $\{e^{i\theta}y\}_{\theta \in [0, 2\pi]}$ is also a point whose density for T_∞ equals that of y . In other words, if y is in the support of T_∞ , so is the whole fiber $\{e^{i\theta}y\}_{\theta \in [0, 2\pi]}$. Denote by $H : S^{2n+1} \rightarrow \mathbb{CP}^n$ the standard Hopf projection. Then, in order to prove lemma 2.1, we actually need to show the following

Proposition 2. *Let T be a positive- (p, p) integral cycle. Let $\{x_m\}$ be a sequence of points such that $x_m \in \mathcal{T}$ with $x_m \rightarrow x_0$, $x_m \neq x_0$ and $H\left(\frac{x_m - x_0}{|x_m - x_0|}\right) \rightarrow y \in \mathbb{CP}^n$. Then the support of any tangent cone to T at x_0 must contain the Hopf circle $H^{-1}(y)$.*

This proposition will be proved by employing an algebraic blow up of the semi-calibrated current T around x_0 . We now shortly recall the notations and the construction, which is developed in more detail in [4].

Since tangent cones to T at a point x_0 are a local issue, we can assume straight from the beginning to work in the unit geodesic ball, in normal coordinates centered at x_0 ; for this purpose it is enough to start with the current T already dilated enough around x_0 . Always up to a dilation, without loss of generality we can actually start with the following situation.

T is a Ω -positive normal cycle in the ball $B_2^{2n+2}(0)$, the coordinates are normal with respect to the origin, J is the standard complex structure at the origin, ω is the standard symplectic form at the origin, $\|\omega - \omega_0\|_{C^{2,\nu}}(B_2^{2n+2})$, $\|\Omega - \Omega_0\|_{C^{2,\nu}}(B_2^{2n+2})$ and $\|J - J_0\|_{C^{2,\nu}}(B_2^{2n+2})$ are small enough.

How to blow up the origin. We shall be using standard coordinates (z_0, z_1, \dots, z_n) in $B_2^{2n+2}(0) \subset \mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$ and the following notations as in [4]:

$$\mathcal{S}_\varepsilon := \{(z_0, z_1, \dots, z_n) \in B_{1+\varepsilon}^{2n+2} \subset \mathbb{C}^{n+1} : |(z_1, \dots, z_n)| < (1 + \varepsilon)|z_0|\},$$

$$\mathcal{V}_\varepsilon \subset \mathbb{CP}^n, \quad \mathcal{V}_\varepsilon := \{[z_0, z_1, \dots, z_n] : |(z_1, \dots, z_n)| < (1 + \varepsilon)|z_0|\}.$$

for $X = [Z_1, \dots, Z_{n+1}] \in \mathcal{V}_\varepsilon$ D^X is the “straight” 2-plane

made of all points $\{\zeta(Z_1, \dots, Z_{n+1}) : \zeta \in \mathbb{C}\}$.

We also write \mathcal{S} for \mathcal{S}_0 and \mathcal{V} for \mathcal{V}_0 .

As shown in section 3 of [4], by constructing (via a fixed point theorem) a *pseudo-holomorphic polar foliation* we can produce an appropriate diffeomorphism

$$\Psi : \mathcal{S}_\varepsilon \rightarrow \Psi(\mathcal{S}_\varepsilon) \approx \mathcal{S}_\varepsilon, \quad (2)$$

which is close to the identity on \mathcal{S}_ε , and which (by pulling-back the problem via Ψ) allows us to make an extra assumption on the almost complex structure J : namely the “straight 2-planes” D^X are J -pseudo holomorphic for all $X \in \mathcal{V}_\varepsilon$. Figure 1 in [4] visually explains the behaviour of Ψ .

With this extra assumption on J , we can proceed to blow-up the origin of \mathbb{C}^{n+1} as follows.

Reminder: algebraic blow-up (from *symplectic* or *algebraic geometry*, see [14]). Define $\tilde{\mathbb{C}}^{n+1}$ to be the submanifold of $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ made of the pairs $(\ell, (z_0, \dots, z_n))$ such that $(z_0, \dots, z_n) \in \ell$.

Denote by I_0 the complex structure that $\tilde{\mathbb{C}}^{n+1}$ inherits from $\mathbb{CP}^n \times \mathbb{C}^{n+1}$. Let $\Phi : \tilde{\mathbb{C}}^{n+1} \rightarrow \mathbb{C}^{n+1}$ be the projection map $(\ell, (z_0, \dots, z_n)) \rightarrow (z_0, \dots, z_n)$. This map is holomorphic for the standard complex structures J_0 on \mathbb{C}^{n+1} and I_0 on $\tilde{\mathbb{C}}^{n+1}$ and is a diffeomorphism between $\tilde{\mathbb{C}}^{n+1} \setminus (\mathbb{CP}^n \times \{0\})$ and $\mathbb{C}^{n+1} \setminus \{0\}$. Moreover the inverse image of $\{0\}$ is $\mathbb{CP}^n \times \{0\}$.

We will endow $\tilde{\mathbb{C}}^{n+1}$ with other almost complex structures, different from I_0 , so $\tilde{\mathbb{C}}^{n+1}$ should be thought of just as an oriented manifold and the structure on it will be specified in every instance. The transformation Φ^{-1} (called *proper transform*) sends the point $0 \neq (z_0, \dots, z_n) \in \mathbb{C}^{n+1}$ to the point $([\ell, (z_0, \dots, z_n)], (z_0, \dots, z_n)) \in \tilde{\mathbb{C}}^{n+1} \subset \mathbb{CP}^n \times \mathbb{C}^{n+1}$.

We will keep using the same letters Φ and Φ^{-1} to denote the same maps restricted to

$$\mathcal{S}_\varepsilon \subset B^{2n+2} \subset \mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1} \text{ and } \mathcal{A}_\varepsilon := (\Phi^{-1})(\mathcal{S}_\varepsilon) \subset \tilde{\mathbb{C}}^{n+1},$$

also when we look at these spaces just as oriented manifolds (not complex ones). We will make use of the notation

$$\mathcal{S}_{\varepsilon, \rho} := \mathcal{S}_\varepsilon \cap B_\rho^{2n+2} \subset \mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1} \text{ and } \mathcal{A}_{\varepsilon, \rho} := \Phi^{-1}(\mathcal{S}_{\varepsilon, \rho}) \subset \tilde{\mathbb{C}}^{n+1}.$$

Let g_0 denote the standard metric⁶ on \mathcal{A}_ε as a subset of $\tilde{\mathbb{C}}^{n+1} \subset \mathbb{CP}^n \times \mathbb{C}^{n+1}$ and ϑ_0 be the standard symplectic form on \mathcal{A}_ε , uniquely defined by $\vartheta_0(\cdot, \cdot) := g_0(\cdot, -I_0 \cdot)$.

⁶The standard metric on $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ is the product of the Fubini-Study metric on \mathbb{CP}^n and the flat metric on \mathbb{C}^{n+1} .

Define on $\mathcal{A}_\epsilon \setminus (\mathbb{CP}^n \times \{0\})$:

- the almost complex structure $I := \Phi^* J$, $I(\cdot) := (\Phi^{-1})_* J \Phi_*(\cdot)$,
- the metric $\mathbf{g}(\cdot, \cdot) := g_0(\cdot, \cdot) + g_0(I\cdot, I\cdot)$,
- the non-degenerate two-form $\vartheta(\cdot, \cdot) := \mathbf{g}(\cdot, -I\cdot)$.

The triple $(I, \mathbf{g}, \vartheta)$ makes $\mathcal{A}_\epsilon \setminus (\mathbb{CP}^n \times \{0\})$ an almost complex manifold and from [4] we have

Lemma 3.2. *The triple $(I, \mathbf{g}, \vartheta)$ extended to \mathcal{A}_ϵ by setting it to be (I_0, g_0, ϑ_0) on $\mathcal{V}_\epsilon \times \{0\}$ is Lipschitz continuous on \mathcal{A}_ϵ and fulfils*

$$|I - I_0|(\cdot) \leq c \text{dist}_{g_0}(\cdot, \mathbb{CP}^n \times \{0\}),$$

$$|\mathbf{g} - g_0|(\cdot) \leq c \text{dist}_{g_0}(\cdot, \mathbb{CP}^n \times \{0\}),$$

$$|\vartheta - \vartheta_0|(\cdot) \leq c \text{dist}_{g_0}(\cdot, \mathbb{CP}^n \times \{0\}),$$

for some constant $c > 0$, which is $o(1)$ of $|J - J_0|$.

Set $\Theta := \frac{1}{p!} \vartheta^p$ on \mathcal{A}_ϵ . The aim is now to translate our original problem to the new space $(\mathcal{A}, I, \mathbf{g}, \vartheta)$. For any $\rho > 0$ we can take the proper transform of $T\mathbb{L}(\mathcal{S}_\epsilon \setminus \mathcal{S}_{\epsilon, \rho})$, since (Φ^{-1}) is a diffeomorphism away from the origin:

$$P_\rho := (\Phi^{-1})_*(T\mathbb{L}(\mathcal{S}_\epsilon \setminus \mathcal{S}_{\epsilon, \rho})).$$

The current P_ρ is clearly positive- (p, p) in $(\mathcal{A}, I, \mathbf{g}, \vartheta)$. What happens when $\rho \rightarrow 0$? Here is the answer.

Lemma 3.3. *(i) The current $P := \lim_{\rho \rightarrow 0} P_\rho = \lim_{\rho \rightarrow 0} (\Phi^{-1})_*(T\mathbb{L}(\mathcal{S}_\epsilon \setminus \mathcal{S}_{\epsilon, \rho}))$ is well-defined as the limit of currents of equibounded mass. The mass of P (both with respect to \mathbf{g} and to g_0) is bounded by a dimensional constant C times the mass of T .*

(ii) P is an integral cycle in \mathcal{A} and it is semi-calibrated by Θ .

The proof is analog to the one in [4], but we need to take care of the fact that the dimension of the current is higher. With the same notations as in [4], for any ρ consider the dilation $\lambda_\rho(\cdot) := \frac{\cdot}{\rho}$, sending B_ρ to B_1 , and the map

$$\Lambda_\rho : \mathcal{A}_\rho \rightarrow \mathcal{A}, \quad \Lambda_\rho := \Phi^{-1} \circ \lambda_\rho \circ \Phi, \quad (3)$$

which in the coordinates of $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ (the ambient space in which \mathcal{A} is embedded) reads $\Lambda_\rho(\ell, z) = \left(\ell, \frac{z}{\rho}\right)$.

proof of lemma 3.3 (i). Recall that we have a uniform bound $M(T_{0,r}) \leq K$, for a constant K independent of r .

Each $P_\rho = (\Phi^{-1})_*(T\mathbb{L}(\mathcal{S}_\varepsilon \setminus \mathcal{S}_{\varepsilon,\rho}))$ is Θ -positive by construction, so $M(P_\rho) = P_\rho(\Theta)$, where the mass is computed here with respect to \mathbf{g} . In order to study the limit as $\rho \rightarrow 0$, it is enough to look at an arbitrary sequence $\rho_n \rightarrow 0$ and prove that P_{ρ_n} have equibounded masses and thus converge to a limit P , which must then be the limit of the whole family P_ρ .

1st step: choice of the sequence. Denote by $\langle T, |z| = r \rangle$ the slice of T with the sphere ∂B_r . Choose ρ_k so to ensure

- (i) $T_{\rho_k} \rightharpoonup T_\infty$ in \mathcal{S} for a certain cone T_∞ ,
- (ii) $M(\langle T_{\rho_k}, |z| = 1 \rangle)$ are equibounded by $4K$,
or, equivalently, $M(\langle T, |z| = \rho_k \rangle) \leq 4K\rho_k^{2p-1}$.

This is just like step 1 of lemma 4.2 in [4].

2nd step: uniform bound on the masses. We use in \mathcal{A} standard coordinates inherited from $\mathbb{CP}^n \times \mathbb{C}^{n+1}$, i.e. we have $2n$ horizontal variables (from \mathbb{CP}^n) and $2n + 2$ vertical variables. We want to estimate $M(P_\rho) = P_\rho(\Theta) = P_\rho(\Theta_0) + P_\rho(\Theta - \Theta_0)$, where $\Theta_0 := \frac{1}{p!}\vartheta_0^p$ on \mathcal{A} . From lemma 3.2 we get that $|\Theta - \Theta_0|(p) < c \operatorname{dist}_{g_0}(p, \mathbb{CP}^n \times \{0\})$ (we keep denoting the constant by c , although it is generally different than the one in lemma 3.2; what is important is that it is still controlled by $|J - J_0|$).

Recall that $\Theta_0 := \frac{1}{p!}\vartheta_0^p$; the domain \mathcal{S}_ε is a product $\mathbb{CP}^n \times \mathbb{C}$ and therefore the standard form ϑ_0 is $\vartheta_{\mathbb{CP}^n} + \vartheta_{\mathbb{C}^n}$, where $\vartheta_{\mathbb{CP}^n}$ is the standard symplectic form on \mathbb{CP}^n extended to \mathcal{S}_ε (so independent of the two “vertical variables”) and $\vartheta_{\mathbb{C}^n}$ is the symplectic two-form on \mathbb{C}^n , extended to \mathcal{S}_ε (so independent of the “horizontal variables”).

Taking the p -th wedge power we get

$$\Theta_0 := \frac{1}{p!}\vartheta_0^p = \frac{1}{p!}(\vartheta_{\mathbb{CP}^n})^p + \sum_{m=1}^p (\vartheta_{\mathbb{CP}^n})^{p-m}(\vartheta_{\mathbb{C}})^m. \quad (4)$$

Let us first estimate $|P_\rho((\vartheta_{\mathbb{CP}^n})^p)|$.

$(\Phi^{-1})^*(\vartheta_{\mathbb{CP}^n}) = \partial\bar{\partial}\log\left(1 + \sum_{j=1}^n \frac{|z_j|^2}{|z_0|^2}\right)$ in the domain \mathcal{S}_ε where $z_0 \neq 0$.

In particular

$$(\Phi^{-1})^*(\vartheta_{\mathbb{CP}^n}) = d\eta, \text{ where } \eta = \frac{1}{2} \left(\bar{\partial}\log\left(1 + \sum_{j=1}^n \frac{|z_j|^2}{|z_0|^2}\right) - \partial\log\left(1 + \sum_{j=1}^n \frac{|z_j|^2}{|z_0|^2}\right) \right).$$

We thus have

$$\begin{aligned}
P_\rho((\vartheta_{\mathbb{CP}^n})^p) &= (T\mathbf{L}(\mathcal{S}_\varepsilon \setminus \mathcal{S}_{\varepsilon,\rho}))((\Phi^{-1})^*(\vartheta_{\mathbb{CP}^n})^p) = \frac{1}{p!} (T\mathbf{L}(\mathcal{S}_\varepsilon \setminus \mathcal{S}_{\varepsilon,\rho})) (d\eta)^p = \\
&= \frac{1}{p!} \partial [T\mathbf{L}(\mathcal{S}_\varepsilon \setminus \mathcal{S}_{\varepsilon,\rho})] [\eta \wedge (d\eta)^{p-1}].
\end{aligned}$$

The boundary of $T\mathbf{L}(\mathcal{S}_\varepsilon \setminus \mathcal{S}_{\varepsilon,\rho})$ is made of three portions: two live in the spheres ∂B_1 and ∂B_ρ and the third one is given by the slice with a hypersurface of the form $\sum_{j=1}^n \frac{|z_j|^2}{|z_0|^2} = \text{const}$. The explicit form of η then implies that this latter portion of boundary has zero action on $\eta \wedge (d\eta)^{p-1}$. We can thus write

$$P_\rho((\vartheta_{\mathbb{CP}^n})^p) = \langle (T\mathbf{L}\mathcal{S}_\varepsilon), r=1 \rangle [\eta \wedge (d\eta)^{p-1}] - \langle (T\mathbf{L}\mathcal{S}_\varepsilon), r=\rho \rangle [\eta \wedge (d\eta)^{p-1}].$$

Now observe the comass of $\eta \wedge (d\eta)^{p-1}$ on the spheres ∂B_ρ . The comasses are equivalent up to a universal constant $C(p, n)$ to the maximum modulus of the coefficients.

For η we can explicitly compute $|\eta| \leq \frac{K}{\rho}$ and $|d\eta| \leq \frac{K}{\rho^2}$ on ∂B_ρ .

Now we focus on the sequence ρ_k chosen in step 1, for which it holds $M \langle (T\mathbf{L}\mathcal{S}_\varepsilon), r=\rho_k \rangle \leq 4K\rho_k^{2p-1}$. We thus get $P_{\rho_k}(\Theta_0) \leq K(p, n)$ independently of ρ_k .

We pass now to estimating the other wedge products $\sum_{m=1}^p (\vartheta_{\mathbb{CP}^n})^{p-m} (\vartheta_{\mathbb{C}^n})^m$ left from (4); the key observation is that $(\Phi^{-1})^*(\vartheta_{\mathbb{C}^n})$ has unit comass, and therefore the forms $(\Phi^{-1})^*((\vartheta_{\mathbb{CP}^n})^{p-m} (\vartheta_{\mathbb{C}^n})^m)$ for $m \in \{1, \dots, p\}$ all have comasses bounded by $\frac{K}{\rho^{2p-2}}$, where ρ is the distance from the origin and K is a universal constant. We then argue using a dyadic decomposition for the estimate on $|P_\rho(\Theta - \Theta_0)|$, as follows.

Break up $\mathcal{S}_\varepsilon = \cup_{j=0}^\infty A_j$, where $A_j = \mathcal{S}_\varepsilon \cap (B_{\frac{1}{2^j}} \setminus B_{\frac{1}{2^{j+1}}})$. It holds $M(T\mathbf{L}A_j) \leq K \frac{1}{2^{2pj}}$. On the other hand, in the same annulus A_j , the comass of the form $(\Phi^{-1})^*(\sum_{m=1}^p (\vartheta_{\mathbb{CP}^n})^{p-m} (\vartheta_{\mathbb{C}^n})^m)$ is $\leq K(p, n) 2^{2(p-1)(j+1)}$, for a constant $K(p, n)$ which only depends on the dimensions involved.

Therefore summing on all j 's we can bound

$$\begin{aligned}
\left| P_\rho \left(\sum_{m=1}^p (\vartheta_{\mathbb{CP}^n})^{p-m} (\vartheta_{\mathbb{C}^n})^m \right) \right| &= \left| (T\mathbf{L}\mathcal{S}_\varepsilon) \left((\Phi^{-1})^* \left(\sum_{m=1}^p (\vartheta_{\mathbb{CP}^n})^{p-m} (\vartheta_{\mathbb{C}^n})^m \right) \right) \right| \leq \\
&\leq K(p, n) \sum_{j=0}^\infty 2^{2(p-1)(j+1)} \frac{1}{2^{2pj}} = K(p, n) \sum_{j=0}^\infty 2^{2p-2-2j} < \infty,
\end{aligned}$$

therefore $|P_\rho(\sum_{m=1}^p (\vartheta_{\mathbb{CP}^n})^{p-m} (\vartheta_{\mathbb{C}^n})^m)|$ is also equibounded independently of ρ .

To conclude the proof of part (i) of lemma 3.3, we must still prove that $|P_\rho(\Theta - \Theta_0)|$ is finite. Thanks to the Lipschitz control on $\vartheta - \vartheta_0$, which also yields $|\Theta - \Theta_0|(\cdot) \leq c \text{dist}_{g_0}(\cdot, \mathbb{CP}^n \times \{0\})$, the form $(\Phi^{-1})^*(\Theta - \Theta_0)$ in \mathcal{S}_ε has comass $\leq \frac{K}{\rho^{2p-1}}$, where ρ is the distance from the origin. Arguing with a dyadic decomposition as done above, we find that also $|P_\rho(\Theta - \Theta_0)|$ is bounded independently of ρ .

We have thus obtained that $M(P_{\rho_n})$ are uniformly bounded as $\rho_n \rightarrow 0$ and therefore there exists a current P in \mathcal{A}_ε such that $P_\rho \rightharpoonup P$. \square

proof of lemma 3.3 (ii). *Step 1.* Let us think of P and $P_\rho := (\Phi^{-1})_*(T\mathbb{L}(\mathcal{S} \setminus \mathcal{S}_\rho))$ as currents in the open set \mathcal{A} in the manifold $\tilde{\mathbb{C}}^{n+1}$. Given a sequence $\rho_k \rightarrow 0$, we want to observe the boundaries ∂P_{ρ_k} . Up to a subsequence we may assume that ρ_k is such that $T_{0,\rho_k} \rightharpoonup T_\infty$ for a certain cone. Then the boundaries ∂P_{ρ_k} satisfy, as $k \rightarrow \infty$, by the definition (3) of Λ_{ρ_k} :

$$(\Lambda_{\rho_k})_*(\partial P_{\rho_k}) = -(\Phi^{-1})_* \langle T_{0,\rho_k}, |z| = 1 \rangle \rightharpoonup -(\Phi^{-1})_* \langle T_\infty, |z| = 1 \rangle. \quad (5)$$

Recall that we are viewing P_{ρ_k} as currents in the open set \mathcal{A} , so also $T\mathbb{L}(\mathcal{S} \setminus \mathcal{S}_\rho)$ should be thought of as a current in the open set \mathcal{S} : this is why the only boundary comes from the slice of T with $|z| = \rho_k$.

Moreover if the sequence is chosen (and we will do so) as in the 1st step of the proof of lemma 3.3 (i), then $(\Lambda_{\rho_k})_*(\partial P_{\rho_k})$ have equibounded masses, since so do the $\partial(T_{0,\rho_k})$'s and Φ^{-1} is a diffeomorphism on ∂B_1 .

The current T_∞ has a special form: it is a positive- (p, p) -cone, so the $(2p-1)$ -current $\langle T_\infty, |z| = 1 \rangle$ has an associated $(2p-1)$ -vector field that always contains the direction tangent to the Hopf fibers⁷ of S^{2n+1} .

Step 2. We want to show that P is a cycle in \mathcal{A} , i.e. that $\partial P_{\rho_k} \rightharpoonup 0$ as $n \rightarrow \infty$. The boundary in the limit could possibly appear on $\mathbb{CP}^n \times \{0\}$ and we can exclude that as follows.

Let α be a $(2p-1)$ -form of comass one with compact support in \mathcal{A} and let us prove that $\partial P_{\rho_k}(\alpha) \rightarrow 0$. Since \mathcal{A} is a submanifold in $\mathbb{CP}^n \times \mathbb{C}^{n+1}$, we can extend α to be a form in $\mathbb{CP}^n \times \mathbb{C}^{n+1}$. Let us write, using horizontal coordinates $\{t_j\}_{j=1}^{2n}$ on \mathbb{CP}^n and vertical ones $\{s_j\}_{j=1}^{2n+2}$ for \mathbb{C}^{n+1} ,

$$\alpha = \alpha_h + \alpha_{v1} + \alpha_{v2} + \dots \alpha_{v(2p-1)},$$

where α_h is a form only in the dt_j 's, and each α_{vj} (for $j = 1, 2, \dots, (2p-1)$) contains wedge products of $(2p-1-j)$ of the dt_j 's and j of the ds_j 's. Rewrite, viewing P_{ρ_n} as currents in $\mathbb{CP}^n \times \mathbb{C}^{n+1}$,

⁷The Hopf fibration is defined by the projection $H : S^{2n+1} \subset \mathbb{C}^{n+1} \rightarrow \mathbb{CP}^n$, $H(z_0, \dots, z_n) = [z_0, \dots, z_n]$. The Hopf fibers $H^{-1}(p)$ for $p \in \mathbb{CP}^n$ are maximal circles in S^{2n+1} , namely the links of complex lines of \mathbb{C}^{n+1} with the sphere.

$$\partial P_{\rho_k}(\alpha) = [(\Lambda_{\rho_k})_*(\partial P_{\rho_k})] (\Lambda_{\rho_k}^{-1})^* \alpha .$$

The map $\Lambda_{\rho_k}^{-1}$ is expressed in our coordinates by $(t_1, \dots, t_{2n}, s_1, \dots, s_n) \rightarrow (t_1, \dots, t_{2n}, \rho_k s_1, \dots, \rho_k s_{2n+2})$, therefore

$$(\Lambda_{\rho_k}^{-1})^* \alpha = \alpha_h^n + \alpha_{v_1}^n + \alpha_{v_2}^n + \dots \alpha_{v(2p-1)}^n,$$

where the decomposition is as above and with $\|\alpha_h^n\|^* \approx \|\alpha_h\|^*$ and $\|\alpha_{v_j}^n\|^* \lesssim (\rho_k)^j \|\alpha_v\|^*$. The signs \approx and \lesssim mean respectively equality and inequality of the comasses up to a dimensional constant, so independently of the index n of the sequence.

As $k \rightarrow \infty$ it holds $\alpha_h^k \rightarrow \alpha_h^\infty$ in some C^ℓ -norm, where $\|\alpha_h^\infty\|^* \lesssim 1$ and α_h^∞ is a form in the dt_j 's⁸. We can write

$$\left| [(\Lambda_{\rho_k})_*(\partial P_{\rho_k})] (\alpha_h^k) \right| \leq \left| [(\Lambda_{\rho_k})_*(\partial P_{\rho_k})] (\alpha_h^k - \alpha_h^\infty) \right| + \left| [(\Lambda_{\rho_k})_*(\partial P_{\rho_k})] (\alpha_h^\infty) \right|$$

and both terms on the r.h.s. go to 0. The first, since $M((\Lambda_{\rho_k})_*(\partial P_{\rho_k}))$ are equibounded and $|\alpha_h^k - \alpha_h^\infty| \rightarrow 0$; the second because we can use (5) and $(\Phi^{-1})_* \partial(T_\infty)$ has zero action on a form that only has the dt_j 's components, as remarked in step 1.

Moreover

$$\left| [(\Lambda_{\rho_k})_*(\partial P_{\rho_k})] (\alpha_{v_j}^k) \right| \rightarrow 0$$

for any $j \in \{1, 2, \dots, (2p-1)\}$, because the currents $(\Lambda_{\rho_k})_*(\partial P_{\rho_k}) = -(\Phi^{-1})_* \langle T_{0, \rho_k}, |z| = 1 \rangle$ have equibounded masses by the choice of ρ_k , while the comasses $\|\alpha_{v_j}^k\|^* \lesssim (\rho_k)^j \|\alpha_v\|^*$ go to 0.

Therefore no boundary appears in the limit and P is an integral cycle in \mathcal{A} . The fact that it is semi-calibrated by Θ follows easily by the fact that so are the currents P_ρ , as remarked just before lemma 3.3. \square

Summarizing, we are now able to take the proper transform of an integral cycle T semi-calibrated by Ω in $\mathcal{S}_\varepsilon \subset B_1^{2n+2}$ and get an integral cycle P in \mathcal{A}_ε that is semi-calibrated by Θ , where the semicalibration Θ is Lipschitz (and actually smooth away from $\mathbb{CP}^n \times \{0\}$). Therefore the almost monotonicity formula holds true for P in \mathcal{A}_ε .

⁸More precisely α_h^∞ coincides with the restriction of α_h to $\mathbb{CP}^n \times \{0\}$, extended to $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ independently of the s_j variables.

4 Proof of the results

With the assumptions in proposition 2 we have to observe a sequence $(\lambda_{r_n})_* T$ as $r_n \rightarrow 0$. Recall that we have assumed (see (2)) that the “straight” 2-planes D^X are pseudo-holomorphic for J .

Take any converging sequence $T_{0,r_n} := (\lambda_{r_n})_* T \rightarrow T_\infty$ for $r_n \rightarrow 0$. Take the proper transform of each T_{0,r_n} and denote it by P_n . Remark that P_n is a Θ_n -semi-calibrated cycle, for a semicalibration Θ_n that is smooth away from $\mathbb{CP}^n \times \{0\}$ and Lipschitz-continuous, with $|\Theta_n - \Theta_0| < c_n \text{dist}_{g_0}(\cdot, \mathbb{CP}^n \times \{0\})$ and the constants c_n go to 0 as $n \rightarrow \infty$ (lemma 3.2).

From lemma 3.3, the masses of P_n are uniformly bounded in n , since so are the masses of T_{0,r_n} (by almost-monotonicity).

So by compactness, up to a subsequence that we do not relabel, we can assume $P_n \rightarrow P_\infty$ for a normal cycle P_∞ .

Lemma 4.1. *P_∞ is a Θ_0 -semi-calibrated cycle; more precisely it is the proper transform of T_∞ .*

Proof. Θ_0 -positiveness follows straight from the Θ_n -positiveness of P_n and $|\Theta_n - \Theta_0| < c_n \text{dist}_{g_0}(\cdot, \mathbb{CP}^n \times \{0\})$, $c_n \rightarrow 0$.

The proper transform is a diffeomorphism away from the origin, thus

$$P_\infty \llcorner (\mathcal{A} \setminus \mathcal{A}_\rho) = \lim_n (\Phi^{-1})_* T_{0,r_n} \llcorner (\mathcal{S}_\varepsilon \setminus \mathcal{S}_{\varepsilon,\rho}) = (\Phi^{-1})_* T_\infty \llcorner (\mathcal{S}_\varepsilon \setminus \mathcal{S}_{\varepsilon,\rho}),$$

so in order to conclude that P_∞ is the proper transform of $(\Phi^{-1})_* T_\infty$ we only need to show $P_\infty = \lim_{\rho \rightarrow 0} P_\infty \llcorner (\mathcal{A} \setminus \mathcal{A}_\rho)$, i.e. that $M(P_\infty \llcorner \mathcal{A}_\rho) \rightarrow 0$ as $\rho \rightarrow 0$.

Recall that $\vartheta_0 = \vartheta_{\mathbb{CP}^n} + \vartheta_{\mathbb{C}^n}$; we want to estimate $M(P_\infty \llcorner \mathcal{A}_\rho) = (P_\infty \llcorner \mathcal{A}_\rho)(\Theta_0) = \lim_{n \rightarrow \infty} (P_n \llcorner \mathcal{A}_\rho)(\Theta_0)$. Write

$$(P_n \llcorner \mathcal{A}_\rho)(\Theta_0) = \frac{1}{p!} (P_n \llcorner \mathcal{A}_\rho)((\vartheta_{\mathbb{CP}^n})^p) + \frac{1}{p!} (P_n \llcorner \mathcal{A}_\rho) \left(\sum_{k=1}^p (\vartheta_{\mathbb{CP}^n})^{p-k} (\vartheta_{\mathbb{C}^n})^k \right). \quad (6)$$

The second term on the r.h.s. is bounded as follows:

$$(P_n \llcorner \mathcal{A}_\rho) \left(\sum_{k=1}^p (\vartheta_{\mathbb{CP}^n})^{p-k} (\vartheta_{\mathbb{C}^n})^k \right) = (\Lambda_\rho)_* (P_n \llcorner \mathcal{A}_\rho) \left((\Lambda_r^{-1})^* \left(\sum_{k=1}^p (\vartheta_{\mathbb{CP}^n})^{p-k} (\vartheta_{\mathbb{C}^n})^k \right) \right).$$

The current $(\Lambda_\rho)_* (P_n \llcorner \mathcal{A}_\rho)$ is the proper transform of $T_{0,\rho r_n}$, therefore $M((\Lambda_\rho)_* (P_n \llcorner \mathcal{A}_\rho)) \leq K$ independently of n ; the form in brackets has comass bounded by ρ^2 . Altogether

$$(P_n \llcorner \mathcal{A}_\rho) \left(\sum_{k=1}^p (\vartheta_{\mathbb{CP}^n})^{p-k} (\vartheta_{\mathbb{C}^n})^k \right) \leq C(p, n) \rho^2.$$

To bound the first term on the r.h.s. of (6), let P be the proper transform of T ; using $(\Lambda_r)^*(\vartheta_{\mathbb{CP}^n})^p = (\vartheta_{\mathbb{CP}^n})^p$ we can write

$$(P_n \llcorner \mathcal{A}_\rho)((\vartheta_{\mathbb{CP}^n})^p) = (P \llcorner \mathcal{A}_{r_n \rho})(\vartheta_{\mathbb{CP}^n})^p \leq M (P \llcorner \mathcal{A}_{r_n \rho}) \leq M (P \llcorner \mathcal{A}_\rho)$$

so it goes to 0 as $\rho \rightarrow 0$ (uniformly in n). Summarizing we get that $(P_n \llcorner \mathcal{A}_\rho)(\Theta_0)$ is $o(1)$ of $\rho \rightarrow 0$ uniformly in n ; therefore so is $M(P_\infty \llcorner \mathcal{A}_\rho) = \lim_{n \rightarrow \infty} (P_n \llcorner \mathcal{A}_\rho)(\Theta_0)$ and we can write $P_\infty = \lim_{\rho \rightarrow 0} P_\infty \llcorner (\mathcal{A} \setminus \mathcal{A}_\rho)$. \square

For a positive- (p, p) integral cycle in \mathbb{R}^{2n+2} , as we have already mentioned, each tangent cone is determined by a $(p-1, p-1)$ integral cycle L_∞ in \mathbb{CP}^n that is calibrated by $\frac{(\theta_{\mathbb{CP}^n})^{p-1}}{(p-1)!}$, the normalized power of the Kähler form.

The previous lemma tells us that, for a sequence $r_n \rightarrow 0$ such that $T_{0, r_n} \rightharpoonup T_\infty$, the proper transforms $P_n := (\Phi^{-1})_* T_{0, r_n}$ converge to $(\Phi^{-1})_* T_\infty$, i.e. to the current $L_\infty \times \llbracket D^2 \rrbracket$. Indeed, the fact that a cone T_∞ is radially invariant translates into the fact that its proper transform is invariant under the action of Λ_ρ for any $\rho > 0$.

proof of proposition 2. Let $T_{0, r_n} \rightharpoonup T_\infty$, a possible tangent cone. Let L_∞ be the holomorphic $(p-1, p-1)$ -integral cycle in \mathbb{CP}^n that identifies T_∞ .

If y_0 is a point in the support of L_∞ , then there exists a sequence of points $0 \neq y_j \rightarrow 0$ such that $H\left(\frac{y_j}{|y_j|}\right) \rightarrow y_0$ (where H is the Hopf projection) and radii δ_j such that each ball $B_{\delta_j}(y_j)$ contains a set \mathcal{C}_j of strictly positive \mathcal{H}^{2p} -measure, \mathcal{C}_j is contained in the support of T_{0, r_n} and the balls $B_{\delta_j}(y_j)$ are disjoint.

If we take a different sequence $T_{0, R_n} \rightharpoonup \tilde{T}_\infty$, we still find a sequence of points as before, since $\frac{y_j}{|y_j|}$ is not changed under radial dilations. Take the proper transforms \tilde{P}_n of T_{0, R_n} . The density is preserved almost everywhere under the push-forward via a diffeomorphism. For each \tilde{P}_n we find that, by upper semi-continuity of the density, y_0 is a point of density ≥ 1 for \tilde{P}_n (for all n). Therefore y_0 is of density ≥ 1 for the limit of the currents \tilde{P}_n , i.e. for $\tilde{P}_\infty = (\Phi^{-1})_* \tilde{T}_\infty$: this follows from the monotonicity formula, with an argument as in remark 3.1.

This proves that any point in the support of L_∞ is also in the support of \tilde{L}_∞ , the holomorphic $(p-1, p-1)$ -integral cycle in \mathbb{CP}^n that identifies \tilde{T}_∞ . Since \tilde{T}_∞ is an arbitrary tangent cone this concludes the proposition and therefore lemma 2.1 is proved, i.e. all tangent cones must have the same support. \square

Now we have to make sure that any two links L_∞ and \tilde{L}_∞ (obtained by a blow up with different sequences r_n and R_n) have multiplicities that agree a.e.

The following lemma should be known, but we recall it for sake of completeness.

Lemma 4.2. *Let C be a semicalibrated cycle of dimension m in \mathbb{R}^n . For any x_0 the set of tangent cones to C at x_0 is a closed and connected subset (for the flat distance, which metrizes the weak*-topology on currents of equibounded mass and boundary mass, see [9]).*

proof of lemma 4.2. Let Υ be the set of all possible tangent cones at x_0 . Given a sequence $\{T_k\}_{k=1}^\infty$ in Υ assume that $T_k \rightarrow T$. We want to show that $T \in \Upsilon$.

The assumption $T_k \in \Upsilon$ means that there exists a sequence $r_n^k \rightarrow 0$ such that as $n \rightarrow \infty$ we have $C_{x_0, r_n^k} \rightarrow T_k$. With a diagonal argument we get $T \in \Upsilon$.

Now, to prove connectedness, assume by contradiction that $\Upsilon = \Upsilon_1 \cup \Upsilon_2$, where Υ_1 and Υ_2 are closed and disjoint. Then there exist (in the space of currents) A_1 and A_2 open disjoint neighbourhoods respectively of Υ_1 and Υ_2 . The family of currents $C_{x_0, r}$ ($r \geq 0$) is continuous and should therefore accumulate (as $r \rightarrow 0$) also somewhere outside A_1 and A_2 , contradiction. \square

Lemma 4.3. *Let C_n and C be integral cycles of dimension k calibrated by a k -form ω . Assume that $C_n \rightarrow C$ and that the support \mathcal{C} is the same for all C_n and C and it is compact. Let $\nu_n(x)$ denote the density at x for C_n and $\nu(x)$ analogously the density at x for C (dealing with calibrated cycles, each ν_n or ν is well-defined everywhere).*

Then for every $x \in \mathcal{C}$ it holds $\nu(x) = \lim_{n \rightarrow \infty} \nu_n(x)$.

proof of lemma 4.3. We will achieve the proof in three steps.

Claim (i) for every $x \in \mathcal{C}$ it holds $\nu(x) \geq \limsup_{n \rightarrow \infty} \nu_n(x)$.

This follows from the monotonicity formula. Indeed, let $B_r(x)$ be the ball around x with radius r . By remark 3.1, the weak convergence $C_n \rightarrow C$ yields $M(C_n \llcorner B_r(x)) \rightarrow M(C \llcorner B_r(x))$. By monotonicity we have $M(C_n \llcorner B_r(x)) \geq \alpha_k \nu_n(x) r^k$, thus it must hold, for all $r > 0$,

$$M(C \llcorner B_r(x)) \geq \alpha_k (\limsup_{n \rightarrow \infty} \nu_n(x)) r^k.$$

Since $\nu(x) = \lim_{r \rightarrow 0} \frac{M(C \llcorner B_r(x))}{\alpha_k r^k}$ we can conclude **claim (i)**.

Claim (ii) There exists $L > 0$ such that $\nu_n, \nu \leq L$ everywhere on \mathcal{C} .

For each fixed C_n (resp. C), the density ν_n (resp. ν) is a bounded function: this follows from the facts that the mass is locally finite, the monotonicity formula holds and the density is upper semi-continuous. So, in order to prove claim (ii), assume by contradiction that there exist points $p_n \in \mathcal{C}$ such that $\nu_n(p_n) \uparrow +\infty$ as $n \rightarrow \infty$. Up to a subsequence that we do not relabel we can assume $p_n \rightarrow p$ for a point p in \mathcal{C} . Choose a ball $B_R(p)$ and let $m > 0$ be chosen so that $M(C \llcorner B_R(p)) = \alpha_k \cdot m \cdot R^2$. Choose n_0 large enough so that for all $n \geq n_0$ it holds (i) $\theta_n(p_n) \geq 3m$ and (ii) $|p_n - p| < \frac{R}{10}$. Then consider the balls $B_{\frac{9R}{10}}(p_n)$: they are contained in $B_R(p)$.

By the monotonicity formula applied at p_n for C_n , we get $M(C_n \llcorner B_{\frac{9R}{10}}(p_n)) \geq \alpha_k(3m) \frac{9^2 R^2}{10^2}$ and therefore $M(C_n \llcorner B_R(p)) \geq \alpha_k(3m) \frac{9^2 R^2}{10^2}$. By remark 3.1 we must have $M(C_n \llcorner B_R(p)) \rightarrow M(C \llcorner B_R(p))$, so we can write $M(C \llcorner B_R(p)) \geq \alpha_k(3m) \frac{9^2 R^2}{10^2}$. Since $\frac{3 \cdot 9^2}{10^2} > 1$ we contradicted the assumption $M(C \llcorner B_R(p)) = \alpha_k \cdot m \cdot R^2$.

Claim (iii) for every $x \in \mathcal{C}$ it holds $\nu(x) = \lim_{n \rightarrow \infty} \nu_n(x)$.

It suffices to show, for an arbitrary x , that

$$\nu(x) = \limsup_{n \rightarrow \infty} \nu_n(x). \quad (7)$$

Once this is achieved, choose a subsequence n_k such that $\liminf_{n \rightarrow \infty} \nu_n(x) = \lim_{n_k \rightarrow \infty} \nu_{n_k}(x)$ and apply (7) to the sequence of currents C_{n_k} to show that $\nu(x) = \lim_{n_k \rightarrow \infty} \nu_{n_k}(x) = \liminf_{n \rightarrow \infty} \nu_n(x)$.

Again the main ingredient for (7) is the monotonicity formula, which for an arbitrary $x \in \mathcal{C}$ states

$$\begin{aligned} M(C_n \llcorner B_R(x)) &= \nu_n(x) + \int_{B_R(x)} \frac{|\vec{C}_y \wedge \partial_r|^2}{|y-x|^k} \nu_n(y) d\mathcal{H}^k(y) \llcorner \mathcal{C} \\ M(C \llcorner B_R(x)) &= \nu(x) + \int_{B_R(x)} \frac{|\vec{C}_y \wedge \partial_r|^2}{|y-x|^k} \nu(y) d\mathcal{H}^k(y) \llcorner \mathcal{C}, \end{aligned} \quad (8)$$

where the unit simple k -vector \vec{C}_y represents the approximate tangent to \mathcal{C} at y with the orientation given on C_n and ∂_r is the radial unit vector (with respect to the point x). Therefore the function $\frac{|\vec{C}_y \wedge \partial_r|^2}{|y-x|^k}$ is independent of n (since the underlying \mathcal{C} is always the same and $\pm \vec{C}_y$ both yield the same value for $|\vec{C}_y \wedge \partial_r|$).

Let μ be the finite measure $\frac{|\vec{C}_y \wedge \partial_r|^2}{|y-x|^k} \cdot \mathcal{H}^k(y) \llcorner (\mathcal{C} \cap B_R(x))$. By **claim (ii)** we can apply Fatou's lemma to $L - \nu_n$ and $L - \nu$ to get

$$\int \limsup_n \nu_n(y) d\mu(y) \geq \limsup_n \int \nu_n(y) d\mu(y),$$

which together with **claim (i)** yields

$$\int \nu(y) d\mu(y) \geq \limsup_n \int \nu_n(y) d\mu(y).$$

We can now use this last inequality together with claim (i) and the fact that $M(C_n \llcorner B_R(x)) \rightarrow M(C \llcorner B_R(x))$ to pass to the limit in (8) as $n \rightarrow \infty$: we get that necessarily we have the equality $\nu(x) = \limsup_{n \rightarrow \infty} \nu_n(x)$. \square

proof of theorem 2.1. Let Υ be the family of possible tangent cones to T at x_0 . The elements of Υ are integral $(p-1, p-1)$ -cycles (in \mathbb{CP}^n , the projective space of $\mathbb{C}^{n+1} \equiv T_x \mathcal{M}$) calibrated by $\frac{(\theta_{\mathbb{CP}^n})^{p-1}}{(p-1)!}$ and by lemma 2.1 they all have the same support.

First we are going to prove that there exists a subset $\Upsilon_d \subset \Upsilon$ that is countable and dense in Υ , i.e. Υ is separable. This is achieved as follows.

All currents in Υ are supported on the same rectifiable set \mathcal{C} and they can only differ by the choice of the density function. We can represent $\mathcal{C} \setminus \tilde{\mathcal{C}}$, where $\tilde{\mathcal{C}}$ is a \mathcal{H}^{2p} -null set, as the image of a Borel subset K of \mathbb{R}^{2p} via a Lipschitz map taking values in \mathbb{CP}^n and with Lipschitz constant $\frac{1}{2} \leq L \leq 2$. To obtain this representation, recall (see [15]) that $\mathcal{C} \setminus \tilde{\mathcal{C}}$ is a countable union of disjoint pieces, each piece being the image, via a Lipschitz map with constant close to 1, of a compact subset of \mathbb{R}^{2p} . We can freely change by translation the position of these countably many compact subset of \mathbb{R}^{2p} and make them disjoint, so by denoting their union with K we get the desired representation for $\mathcal{C} \setminus \tilde{\mathcal{C}}$.

For each current in Υ the density on the rectifiable set \mathcal{C} is an L^1 function on $(\mathcal{C}, \mathcal{H}^{2p})$. Using the previous representation of \mathcal{C} , we can record these densities as L^1 functions on \mathbb{R}^{2p} that are zero outside of K . The family Υ therefore yields a family $\{g_a\}_{a \in \Upsilon}$ of such $L^1(\mathbb{R}^{2p})$ functions. Every such L^1 function on \mathbb{R}^{2p} is associated to a current supported on \mathcal{C} .

The family $\{g_a\}$ is compact in L^1 : indeed the L^1 -convergence for a sequence in $\{g_a\}$ yields the (weak*) convergence for the corresponding currents. So $\{g_a\}$ is closed with respect to the L^1 -norm, because Υ is closed with respect to the weak*-topology. Moreover $\{g_a\}$ is bounded in L^1 because $\int_{\mathbb{R}^{2p}} g_a d\mathcal{L}^{2p}$ is comparable (up to a factor 2, recall the condition on L) to the mass of the corresponding current which is fixed for all elements of Υ .

We conclude that, as a compact subspace of the separable normed space $L^1(\mathbb{R}^{2p})$, $\{g_a\}$ is also separable. The corresponding countable set of currents is the desired Υ_d .

Except on a \mathcal{H}^{2p} -null set $\tilde{\mathcal{C}}'$, all points of $\mathcal{C} \setminus \tilde{\mathcal{C}}'$ have integer densities for all currents in Υ_d .

Let now $x \in \mathcal{C} \setminus \tilde{\mathcal{C}}'$ and observe the function F from Υ_d to \mathbb{R} assigning to every current $P \in \Upsilon_d$ the value $F(P) := \nu_P(x)$, where ν_P is the density of

P . By lemma 4.3 the function F is continuous on the metric space Υ_d , but since it is also integer-valued it must be locally constant on Υ_d .

Υ_d is dense in Υ , so for every current $P' \in \Upsilon$ the value $\nu_{P'}(x)$ is also locally constant by lemma 4.3. Since Υ is connected, $\nu_{P'}(x)$ must then be globally constant for $P' \in \Upsilon$. The point $x \in \mathcal{C} \setminus \tilde{\mathcal{C}}'$ was arbitrary, therefore all currents in Υ have a fixed density at all points except on the null set $\tilde{\mathcal{C}}'$ and this makes them equal as currents. A posteriori also the density on $\tilde{\mathcal{C}}'$ is fixed.

We have therefore obtained that Υ is made of one single element and we can conclude the uniqueness theorem 2.1 for tangent cones. \square

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